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LMI-based Adaptive Observers for Nonlinear Systems

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Abstract- *This paper deals with the design of adaptive observers that can estimate both the states and the parameters of a large class of nominal and perturbed nonlinear systems with a regression matrix (i.e. matching matrix with the unknown parameter vector) depending on unknown states. The asymptotic stability of the state and parameter estimate errors is developed in the presence of common persistency of excitation (PE). The observer gain calculus is cast as a linear matrix inequality (LMI) feasibility problem. The appeal of this proven theoretical design is further demonstrated numerically.*

Keywords: Nonlinear System, Perturbed Dynamics, Adaptive Observer, Parameter Estimation, LMI

1. Introduction

The adaptive observer design for linear and nonlinear systems has been widely investigated during the last few decades (Khayati & Zhu, 2011; Zhu & Khayati, 2011; Maatoug *et al.*, 2008; Cho & Rajamani, 1997; Marino & Tomei, 1995) and references cited therein. In (Cho and Rajamani, 1997), the authors have designed a systematic approach for an adaptive observer that estimates the full-state variables for nonlinear dynamics in the presence of uncertain parameters possibly depending on the input and state variables. The stability of the algorithm is guaranteed when at least some of the measured outputs are such that the transfer matrix from the unknown parameters to these outputs is dissipative (Cho & Rajamani, 1997). The design of the observer depends on the ability to solve an LMI problem under a conservative matrix equality refering to "matchning conditions" of the dynamic representation. Even, it has been reported and applied in many other works in the literature (Dimassi *et al.*, 2010; Liu, 2009; Dong & Mei, 2007; Stepanyan & Hovakimyan, 2007; Zhu, 2007), this concept is still hard to achieve within some (but very common) dynamics as discussed in (Zhu and Khayati, 2011).

The design of adaptive observer schemes to estimate jointly the states and the parameters for nonlinear dynamic systems with uncertainties referring more to actual scenarios has been studied (Zhao, *et al.*, 2011; Paesa, *et al.*, 2010; Zemouche & Boutayeba, 2009; Stamnes, *et al.*, 2009; Garimella & Yao, 2003; Marino, *et al.*, 2001). However, there are still problems (other than the matching conditions) in these given adaptive observers, and also, the assumptions are difficult to achieve. In most of these recent works, authors have considered the case of known nonlinear regression matrix of measurable states and inputs (Zhao *et al.*, 2011; Paesa *et al.*, 2010; Garimella & Yao, 2003; Marino *et al.*, 2001). In (Zemouche & Boutayeba, 2009), a unified H_∞ adaptive observer for a class of nonlinear systems is introduced to estimate uncertain parameters in the unmeasured nonlinearities. These nonlinearities must be differentiable so that the differential mean value theorem can be applied. In (Stamnes *et al.*, 2009), the authors have designed a nonlinear adaptive observer for a limited class of nonlinear systems. The proposed adaptation law has been built using nonlinear partial differential equations in known and unknown states. However, the stability of such an observer requires nonlinear time-varying "sector conditions" to be satisfied.

In this paper, a more general form of the adaptive observer scheme discussed in (Zhu and Khayati, 2011) will be extended to nonlinear dynamics with a regression matrix function of both measurable and unmeasurable signals and unknown disturbances. The stability condition of the proposed adaptive observer will be presented using only strict LMIs (Boyd *et al.*, 1994). The proposed design estimating the full states and identifying the unknown parameters for a large class of nonlinear dynamic systems is cast with general conditions that are still feasible and address realistic plants. This paper is organized as follows. In Section 2, we describe the problem statement and assumptions. In

Section 3, we introduce the general form of the nonlinear adaptive observer (NLAO) for the nominal dynamics (with no disturbances), and then, the robust nonlinear adaptive observer (RNAO) for perturbed nonlinear dynamics respectively. Section 4 shows illustrative simulation examples, while Section 5 concludes this work.

2. Statement and Assumptions

2.1. Problem Statement

Consider the nonlinear dynamics with unknown disturbances:

$$\dot{x} = Ax + B_0 f_0(u,y) + B_1 f_1(u,x) + Bf(u,x)\theta + E\omega(t) \quad (1)$$

$$y = Cx + D\omega(t) \quad (2)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^p$ the input vector, $y \in \mathbb{R}^m$ the output, $\theta \in \mathbb{R}^q$ the vector of unknown constant parameters and $\omega(t) \in \mathbb{R}^\eta$ represents unknown disturbances. $A \in \mathbb{R}^{n\times n}$, $B_0 \in \mathbb{R}^{n\times k}$, $B_1 \in \mathbb{R}^{n\times r}$, $B \in \mathbb{R}^{n\times s}$, $E \in \mathbb{R}^{n\times \eta}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times \eta}$ are known constant matrices. $f_0(u,y)$, $f_1(u,x)$ and $f(u,x)$ are nonlinear functions in $\mathbb{R}^k$, $\mathbb{R}^r$ and $\mathbb{R}^{s\times q}$, respectively.

2.2. Assumptions

For the forthcoming design, we consider the following assumptions (Zhu and Khayati, 2011):

A1. The vector of unknown constant parameters θ is bounded, with

$$\|\theta\| \le \alpha \quad (3)$$

A2. $f(u,x)$ is continuously bounded, and both functions f_1 and f are Lipschitz in x, with

$$\|f_1(u,x) - f_1(u,\hat{x})\| \le \beta_1 \|x - \hat{x}\| \quad (4)$$

$$\|f(u,x) - f(u,\hat{x})\| \le \beta \|x - \hat{x}\| \quad (5)$$

A3. The input vector u is of class $\mathcal{C}_1$ (*i.e.* continuous function having continuous first derivatives).

3. Adaptive Observer Design

For the adaptive observer design, we first briefly introduce the design for the case of nominal dynamics *i.e.* disturbance free one; then we extend it to the perturbed case and prove the effectiveness.

3.1. Case 1 – Nominal Dynamics

Given the unperturbed nonlinear model, that is described by (1) and (2) with $D = 0$ and $E = 0$. Consider the following full-order nonlinear observer

$$\dot{\hat{x}} = A\hat{x} + B_0 f_0(u,y) + B_1 f_1(u,\hat{x}) + Bf(u,\hat{x})\hat{\theta} + L(y - C\hat{x}) \quad (6)$$

and adaptation law

$$\dot{\breve{\theta}} = \Gamma \cdot \left[f^T(u,\hat{x}) B^T P(\Phi + \Psi CL) - \bar{f}^T(u,\dot{u},\hat{x}) B^T P \Psi \right](y - C\hat{x}) \quad (7)$$

$$\hat{\theta} = \breve{\theta} + \Gamma \cdot f^T(u,\hat{x}) B^T P \Psi \cdot (y - C\hat{x}) \quad (8)$$

where $\bar{f}(u,\dot{u},\hat{x})$ is the total time derivative of $f(u,\hat{x})$ given by $\bar{f}(u,\dot{u},\hat{x}) = \frac{\partial f}{\partial \hat{x}}(u,\hat{x})\dot{\hat{x}} + \frac{\partial f}{\partial u}(u,\hat{x})\dot{u}$, $\Gamma = \Gamma^T > 0$ matrix of $\mathbb{R}^{q\times q}$, $L \in \mathbb{R}^{n\times m}$, Φ and Ψ matrices of $\mathbb{R}^{n\times m}$.

Proposition 1 – Under assumptions **A1-A3**, if there exist matrices $P = P^T > 0$ in $\mathbb{R}^{n\times n}$ and $W \in \mathbb{R}^{n\times m}$ such that

$$\begin{pmatrix} PA + A^T P - WC - C^T W^T + \delta^2 I_n & P \\ P & -I_n \end{pmatrix} < 0 \quad (9)$$

with $\delta = \beta_1 \|B_1\| + \alpha\beta \|B\| > 0$ and matrices Φ and Ψ of $\mathbb{R}^{n\times m}$ computed from

$$\Psi CB = 0, \quad \Psi CB_1 = 0, \quad \Phi C + \Psi CA = I_n \quad (10)$$

with I_n the identity matrix of $\mathbb{R}^{n\times n}$, then the state estimation error vector $\tilde{x} = x - \hat{x}$ of the NLAO (6)-(8), with the observer gain matrix computed as $L = P^{-1}W$, for the nominal system (1) and (2) tends to zero and the parameter estimate error vector $\tilde{\theta} = \theta - \hat{\theta}$ is radially bounded. In addition, if for some positive scalars α_1, α_2 and t_0 with the inequalities

$$\alpha_1 I_q \le \int_t^{t_0+t} f^T(u,x) B^T B f(u,x) d\tau \le \alpha_2 I_q \quad (11)$$

hold $\forall t$, where I_q is the identity matrix of order q, then both estimate errors $\tilde{x} \to 0$ and $\tilde{\theta} \to 0$ asymptotically as $t \to \infty$. The condition (11) refers to the PE which is very common in the literature (Maatoug *et al.*, 2008; Dong & Mei, 2007; Cho & Rajamani, 1997).

Proof – Let $\tilde{x} = x - \hat{x}$ and $\tilde{\theta} = \theta - \hat{\theta}$ be the state and parameter estimate errors, respectively. From (1), (2) and (6), we derive

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + B_1\left(f_1(u,x) - f_1(u,\hat{x})\right) + \mathrm{B}\big(f(u,x)\theta - f(u,\hat{x})\hat{\theta}\big) \quad (12)$$

Based on assumption **A1**, we have $\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$. From (7) and (8), we derive

$$\begin{aligned}\dot{\tilde{\theta}} &= -\dot{\tilde{\theta}} - \Gamma\overline{f}^T(u,\dot{u},\hat{x})B^T P\Psi C\tilde{x} - \Gamma f^T(u,\hat{x})B^T P\Psi C\dot{\tilde{x}} \\ &= -\Gamma f^T(u,\hat{x})B^T P(\Phi + \Psi CL)C\tilde{x} - \Gamma f^T(u,\hat{x})B^T P\Psi C \cdot \\ &\quad (A - LC)\tilde{x} - \Gamma f^T(u,\hat{x})B^T P\Psi CB_1\left(f_1(u,x) - f_1(u,\hat{x})\right) - \\ &\quad \Gamma f^T(u,\hat{x})B^T P\Psi CB\big(f(u,x)\theta - f(u,\hat{x})\hat{\theta}\big) \end{aligned} \quad (13)$$

By using the conditions (10), (13) reduces to

$$\dot{\tilde{\theta}} = -\Gamma f^T(u,\hat{x})B^T P\tilde{x} \quad (14)$$

Now, to investigate the stability, given $P = P^T > 0$ and $\Gamma = \Gamma^T > 0$, consider the Lyapunov candidate function $V(t) = \tilde{x}^T P\tilde{x} + \tilde{\theta}^T\Gamma^{-1}\tilde{\theta}$ (Khayati and Zhu, 2011). Using (12) and (14), we have

$$\begin{aligned}\dot{V}(t) &= 2\tilde{x}^T P\dot{\tilde{x}} + 2\tilde{\theta}^T\Gamma^{-1}\dot{\tilde{\theta}} \\ &= 2\tilde{x}^T P\Big[(A - LC)\tilde{x} + B_1\left(f_1(u,x) - f_1(u,\hat{x})\right) + \\ &\quad B\big(f(u,x)\theta - f(u,\hat{x})\theta\big)\Big]\end{aligned} \quad (15)$$

Based on the inequalities of assumptions **A1** and **A2**, we have

$$\dot{V} \le \tilde{x}^T\left[P(A - LC) + (A - LC)^T P\right]\tilde{x} + 2\beta_1\|B_1\|\|P\tilde{x}\|\cdot\|\tilde{x}\| + 2\alpha\beta\|B\|\|P\tilde{x}\|\|\tilde{x}\| \quad (16)$$

Using $\delta = \beta_1\|G_1\| + \alpha\beta\|G\| > 0$, we notice the inequality $2\delta\|P\tilde{x}\|\|\tilde{x}\| \le \tilde{x}^T PP\tilde{x} + \delta^2\tilde{x}^T\tilde{x}$, we obtain

$$\dot{V} \le \tilde{x}^T\left[P(A - LC) + (A - LC)^T P + PP + \delta^2 I_n\right]\tilde{x} \quad (17)$$

$\dot{V}$ is negative if

$$P(A - LC) + (A - LC)^T P + PP + \delta^2 I_n < 0 \quad (18)$$

The inequality (18), which is nonlinear in P and L (nonlinearities refer to PP and PL), will be transformed into the LMI (9) by simply applying the Schur complement theorem (Boyd *et al*., 1994) and the change of variable $W = PL$.

Now, from (17) and (18), $\exists\varepsilon > 0$ such that

$$\dot{V} \le -\varepsilon\|\tilde{x}\|^2 \quad (19)$$

This implies $V(t) \in \mathcal{L}_\infty$ (*i.e.* time-functions of finite ∞-norm), and then $\tilde{x} \in \mathcal{L}_\infty$ and $\tilde{\theta} \in \mathcal{L}_\infty$. Integrating (19) leads to

$$V(t) \le V(0) - \varepsilon\int_0^t \|\tilde{x}(\tau)\|^2 d\tau \quad (20)$$

Since $V(0)$ is finite, we obtain $\tilde{x} \in \mathcal{L}_2$ (*i.e.* finite 2-norm vector-function). From (12), we have $\dot{\tilde{x}} \in \mathcal{L}_\infty$. Therefore, by applying theorem 8.4 of (Khalil, 2002) based on Barbalat's Lemma, $\hat{x} \to x$ and $\dot{\tilde{x}} \to 0$. From (12), we have $B\big(f(u,x)\theta - f(u,\hat{x})\hat{\theta}\big) \to 0$. Using the inequality (5) and noting that $\hat{x} \to x$, we have $B\big(f(u,x) - f(u,\hat{x})\big)\theta \to 0$. So, from $\tilde{\theta} = \theta - \hat{\theta} \in \mathcal{L}_\infty$ and θ is constant (*i.e.* assumption A1), we obtain $B\big(f(u,x) - f(u,\hat{x})\big)\hat{\theta} \to 0$ and then $Bf(u,x)\big(\theta - \hat{\theta}\big) \to 0$ as $t \to \infty$ (Dong and Mei 2007). Hence, $f^T(u,x)B^T Bf(u,x)\tilde{\theta} \to 0$ as $t \to \infty$. In the following, we investigate the PE property to lead to $\tilde{\theta}(t) \to 0$. Define $\rho(t_0) = \int_t^{t_0} f^T(u,x)B^T Bf(u,x)d\tau$ (Dong and Mei 2007). Using the integration by parts, we obtain

$$\int_t^{t_0+t} f^T(u,x)B^T Bf(u,x)\tilde{\theta}(\tau)d\tau = \rho(t_0 + t)\cdot\tilde{\theta}(t_0 + t) - \rho(t)\cdot\tilde{\theta}(t) - \int_t^{t_0+t}\rho(\tau)\dot{\tilde{\theta}}(\tau)d\tau \quad (21)$$

As $\rho\ t\ = 0$, we obtain

$$\int_t^{t_0+t} f^T(u,x)B^T Bf(u,x)\tilde{\theta}(\tau)d\tau = \rho(t_0 + t)\cdot\tilde{\theta}(t_0 + t) - \int_t^{t_0+t}\rho(\tau)\dot{\tilde{\theta}}(\tau)d\tau \quad (22)$$

Since $f^T(u,x)B^T Bf(u,x)\tilde{\theta} \to 0$, then for any finite t_0, we have

$$\int_t^{t_0+t} f^T(u,x)B^T Bf(u,x)\tilde{\theta}(\tau)d\tau \to 0 \quad (23)$$

Moreover, since $f(u,x)$ is bounded (see assumption **A2**) and $\tilde{x} \to 0$, from (14), we have $\dot{\tilde{\theta}} \to 0$, and then

$\int_{t}^{t_0+t}\rho(\tau)\dot{\tilde{\theta}}(\tau)d\tau\to 0$ as $t\to\infty$. Thus, from (22), we have $\rho(t_0+t)\cdot\tilde{\theta}(t_0+t)\to 0$, as $t\to\infty$. From the assumption of PE (11), *i.e.* $\alpha_1 I_q\le\rho(t_0+t)\le\alpha_2 I_q\quad\forall t>0$ for positive scalars α_1 and α_2, we obtain $\tilde{\theta}(t_0+t)\to 0$, implying $\tilde{\theta}(t)\to 0$ (Dong & Mei, 2007).

3.2. Case 2 – Perturbed Dynamics

Consider the perturbed dynamics (1) and (2) under assumptions **A1-A3**. We propose the following RNAO scheme

$$\dot{\breve{x}}=\bar{A}\breve{x}+G_0f_0(u,y)+G_1f_1(u,\hat{x})+Gf(u,\hat{x})\hat{\theta}+L\left(Ty-\bar{C}\hat{x}\right)+Ky \tag{24}$$

$$\hat{x}=\breve{x}-\bar{D}y \tag{25}$$

$$\dot{\breve{\theta}}=\Gamma\left[f^T(u,\hat{x})G^TP\left(\Phi+\Psi\bar{C}L\right)-\dot{f}^T(u,\dot{u},\hat{x})G^TP\Psi\right]\cdot\left(Ty-\bar{C}\hat{x}\right) \tag{26}$$

$$\hat{\theta}=\breve{\theta}+\Gamma f^T(u,\hat{x})G^TP\Psi\cdot\left(Ty-\bar{C}\hat{x}\right) \tag{27}$$

where $\dot{f}(u,\dot{u},\hat{x})$ is the total time derivative of $f(u,\hat{x})$; $\bar{A}\in\mathbb{R}^{n\times n}$, $G_0\in\mathbb{R}^{n\times k}$, $G_1\in\mathbb{R}^{n\times r}$, $G\in\mathbb{R}^{n\times s}$, $K\in\mathbb{R}^{n\times m}$, $T\in\mathbb{R}^{\mu\times m}$, $\bar{C}\in\mathbb{R}^{\mu\times n}$, $\bar{D}\in\mathbb{R}^{n\times m}$, $\Phi\in\mathbb{R}^{n\times\mu}$ and $\Psi\in\mathbb{R}^{n\times\mu}$ are constant matrices computed by the following

$$\bar{D}D=0,\quad TD=0,\quad TC=\bar{C} \tag{28}$$

$$G_0=B_0+\bar{D}CB_0,\quad G_1=B_1+\bar{D}CB_1,\quad G=B+\bar{D}CB \tag{29}$$

$$KD=E+\bar{D}CE,\quad \bar{A}=A+\bar{D}CA-\bar{A}\bar{D}C-KC \tag{30}$$

$$\Psi\bar{C}G=0,\quad \Psi\bar{C}G_1=0,\quad \Phi\bar{C}+\Psi\bar{C}\,\bar{A}=I_n \tag{31}$$

with I_n being the identity matrix of $\mathbb{R}^{n\times n}$, $L\in\mathbb{R}^{n\times\mu}$ the observer gain and $\Gamma=\Gamma^T>0$ the adaptation matrix gain of $\mathbb{R}^{q\times q}$.

Proposition 2 – Under assumptions **A1-A3**, if there exist matrices $P=P^T>0$ in $\mathbb{R}^{n\times n}$ and $W\in\mathbb{R}^{n\times m}$ such that (32) holds, then $\tilde{x}=x-\hat{x}\to 0$ and $\tilde{\theta}=\theta-\hat{\theta}$ is radially bounded as $t\to\infty$.

$$\begin{pmatrix} P\bar{A}+\bar{A}^TP-W\bar{C}-\bar{C}^TW^T+\delta^2 I_n & P \\ P & -I_n\end{pmatrix}<0 \tag{32}$$

with $\delta=\beta_1\|G_1\|+\alpha\beta\|G\|>0$. The observer gain matrix is $L=P^{-1}W$. The algebraic equality conditions (28)-(31) complete the computation of the adaptive observer scheme. In addition, if the PE condition

$$\alpha_1 I_q\le\int_{t}^{t_0+t}f^T(u,x)G^TGf(u,x)d\tau\le\alpha_2 I_q,\ \forall t \tag{33}$$

holds for some positive scalars α_1, α_2 and t_0, then the RNAO (24)-(27) for the perturbed dynamics (1) and (2) is asymptotically stable, that is both state and parameter estimates converge asymptotically to their actual values as $t\to\infty$.

Proof – Using $\bar{D}D=0$, the observer output is reduced to $\hat{x}=\breve{x}-\bar{D}Cx$. Its time derivative is then obtained from (1), (2), (24), (25) and $TC=\bar{C}$

$$\begin{aligned}\dot{\hat{x}}=&\bar{A}\hat{x}+\left(\bar{A}\bar{D}C+KC-\bar{D}CA\right)x+L\bar{C}(x-\hat{x})+\left(KD-\bar{D}CE\right)\omega+\\&\left(G_0-\bar{D}CB_0\right)f_0(u,y)+G_1f_1(u,\hat{x})-\bar{D}CB_1f_1(u,x)+\\&Gf(u,\hat{x})\hat{\theta}-\bar{D}CBf(u,x)\theta\end{aligned} \tag{34}$$

Let $\tilde{x}=x-\hat{x}$ be the state estimate error. From (1), (2), (24), (25), we derive

$$\begin{aligned}\dot{\tilde{x}}=&\left(A+\bar{D}CA-\bar{A}\bar{D}C-KC\right)x-\bar{A}\hat{x}-L\bar{C}\tilde{x}+\left(E+\bar{D}CE-KD\right)\omega+\\&\left(B_0+\bar{D}CB_0-G_0\right)f_0(u,y)-G_1f_1(u,\hat{x})+\\&\left(B_1+\bar{D}CB_1\right)f_1(u,x)-Gf(u,\hat{x})\hat{\theta}+\left(B+\bar{D}CB\right)f(u,x)\theta\end{aligned} \tag{35}$$

Using (29) and (30), we obtain

$$\dot{\tilde{x}}=\left(\bar{A}-L\bar{C}\right)\tilde{x}+G_1\left(f_1(u,x)-f_1(u,\hat{x})\right)+G\left(f(u,x)\theta-f(u,\hat{x})\hat{\theta}\right) \tag{36}$$

From (26) and (27), using $TC=\bar{C}$ and the error dynamics (36), we derive

$$\begin{aligned}\dot{\hat{\theta}}=&\dot{\breve{\theta}}+\Gamma\dot{f}^T(u,\hat{x})G^TP\Psi\bar{C}\tilde{x}+\Gamma f^T(u,\hat{x})G^TP\Psi\bar{C}\dot{\tilde{x}}\\=&\Gamma f^T(u,\hat{x})G^TP\left(\Phi+\Psi\bar{C}L\right)\bar{C}\tilde{x}+\Gamma f^T(u,\hat{x})G^TP\Psi\bar{C}\left(\bar{A}-L\bar{C}\right)\tilde{x}+\\&\Gamma f^T(u,\hat{x})G^TP\Psi\bar{C}G_1\left(f_1(u,x)-f_1(u,\hat{x})\right)+\Gamma f^T(u,\hat{x})G^TP\Psi\bar{C}G\cdot\\&\left(f(u,x)\theta-f(u,\hat{x})\hat{\theta}\right)\end{aligned} \tag{37}$$

By using the conditions (31), the dynamics (37) reduces to

$$\dot{\hat{\theta}}=\Gamma f^T(u,\hat{x})G^TP\left(\Phi\bar{C}+\Psi\bar{C}\,\bar{A}\right)\tilde{x}=\Gamma f^T(u,\hat{x})G^TP\tilde{x} \tag{38}$$

Let $\tilde{\theta}=\theta-\hat{\theta}$ be the parameter estimate error. Based on assumption A1, we have the $\dot{\tilde{\theta}}=-\dot{\hat{\theta}}$. Then, we obtain the adaptation error dynamics

$$\dot{\tilde{\theta}} = -\Gamma f^T(u,\hat{x}) G^T P \tilde{x} \quad (39)$$

To investigate the stability of the estimate error dynamics and the proposed LMI feasibility problem, we consider the same Lyapunov function $V(t) = \tilde{x}^T P \tilde{x} + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$ and the PE condition (33), and we follow the same steps of the proof of Proposition 1 shown above.

3.3. Remarks

In the first algorithm, we discuss a nonlinear adaptive observer for disturbance-free dynamics with nonlinear regression matrix function of unknown states. The first contribution of this paper is that the proposed design has the advantage of being applied appropriately for nonlinear dynamics (1)-(2) with the particular property of $CB = 0$; that is, columns of the matrix B lie in the null space of the output matrix C.

Then, we consider the more general case of perturbed nonlinear dynamics, which is the second contribution. The proposed observer in the second scheme decouples the effect of the disturbances from the estimation process. This scheme is expected to improve the accuracy and robustness of estimation when the system is subject to unknown disturbances and noisy measurement. Both schemes represent a generalization of the second order adaptive observer dynamics introduced in (Zhu & Khayati, 2011). The key element of the proposed design is inspired by (Zhao *et al.*, 2011) where the authors have considered only a linear dynamics counterpart with a known regression term.

The matrices of the adaptive observers are computed using equations and LMIs, independently. These independent computations make the design of the NLAO and RNAO feasible provided the common assumptions and the LMI (9) (or (32)) are satisfied. This design is more tractable numerically than the method in (Liu, 2009; Dong & Mei, 2007; Cho & Rajamani, 1997) which needs more effort to solve a set of equalities and inequalities simultanously.

4. Illustrative Examples

In this section, we show three illustrations enhancing the effectiveness of both schemes discussed above. The first example shows a low speed motion with a dynamic friction of unknown states and parameters. In the second example, we consider a disturbed second order mechanical dynamics with nonlinear terms, while the third one represents a third order dynamics including an uncertain nonlinear term.

4.1. Example 1 – Low Speed Motion with Dynamic Friction

Consider a single known mass M at position p

$$M\ddot{p} + F_f = u \quad (40)$$

under the influence of a dynamic friction F_f and an input force u. The friction force F_f is given by the modified LuGre model:

$$\dot{F}_f = \sigma v - \frac{\sigma|v|}{g(v)} F_f = \sigma v - \sigma|v| F_f \times s(v) \quad (41)$$

$$s(v) = \left(\frac{1}{\mu_s} + \frac{h(v)}{\mu_c}\right) \cdot \frac{1}{h(v)+1} \quad (42)$$

where $h(v) = 4\theta_s^{\,2} v^4$ with $v = \dot{p}$ represents the actual velocity. σ is the frictional stiffness. μ_c is the normalized Coulomb friction and μ_s the normalized static friction coefficient. The parameters σ, μ_c and μ_s are unknown. We assume the position p and the velocity v are both measurable, but F_f is unknown and is under the stiffness, Stribeck, static and Coulomb effects in the absence of internal and external damping frictions (Canudas *et al.*, 1995). The term $s(v)$ represents a finite function which is chosen to describe the different friction effects. It replaces the function given in (Canudas *et al.*, 1995):

$$g(v) = F_c + (F_s - F_c) \cdot e^{-\theta_s v^2} \quad (43)$$

In the literature, it was widely proven that the friction parameterization is not limited to (43). Indeed, this term is nonlinear in the unknown parameter. By using (42), the proposed modified LuGre model presents an easy-to-use linear-in-the-parameters form that captures most of the observed static friction phenomena of velocity and the unknown parameters become linearly dependent and thus suitable for any on-line estimation. θ_s denotes the Stribeck time constant and indicates the velocity range in which the Stribeck effect is effective. The friction model is a nonlinear function of θ_s. To prevent further difficulties with the nonlinear estimation technique, an empirical value of θ_s is selected from the literatures (Waiboer *et al.*, 2005; Canudas *et al.*, 1995). The state representation $x_1 = p$, $x_2 = v$, $x_3 = F_f$ and $y = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ defines the system in the state space form (1) and (2) using the model matrices shown in Table 1. For simulation purposes, the parameters characterizing the mechanical system are chosen $M = 0.5\text{kg}$, $\sigma = 10^3\,\text{Nm}^{-1}$, $\mu_c = 0.2\text{N}$, $\mu_s = 1.05\text{N}$ and $\theta_s = 4\times10^4\,\text{s}^2$ respectively. The unknown parameters are $\theta_1 = \sigma$, $\theta_2 = \dfrac{\sigma}{\mu_s}$, and $\theta_3 = \dfrac{\sigma}{\mu_c}$, respectively. To estimate the unknown friction force and parameters, we apply the NLAO design with the computed parameters as shown in Table 1. The estimates of the states are shown in Figures 1-3. The estimates of the parameters are depicted in Figures 4-6. Both the state estimation error and the parameter error converge to zero quickly and accurately.

Table 1. Plant and NLAO parameters in example 1.

Plant	$A=\begin{pmatrix}0&1&0\\0&0&-2\\0&0&0\end{pmatrix}$, $B=\begin{pmatrix}0\\0\\1\end{pmatrix}$, $B_0=\begin{pmatrix}0\\2\\0\end{pmatrix}$, $C=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$, $D\equiv 0$, $E\equiv 0$
NALO	$\Phi=\begin{pmatrix}1&0\\0&0.4\\0&0.5\end{pmatrix}$, $\Psi=\begin{pmatrix}0&0\\0.6&0\\-0.5&-0.5\end{pmatrix}$, $\Gamma=10^3 I_3$, $L=\begin{pmatrix}0.5&25.9\\-26.1&1.0\\5.8&-2.2\end{pmatrix}$, $P=\begin{pmatrix}167.2&0.0&0.0\\0.0&167.2&36.9\\0.0&36.9&167.2\end{pmatrix}$

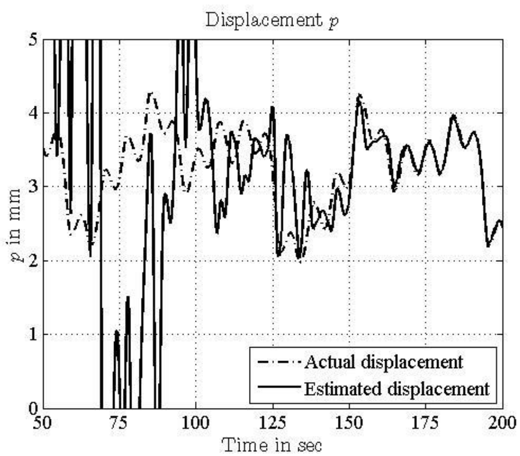

Fig. 1. Position of the low speed motion dynamics.

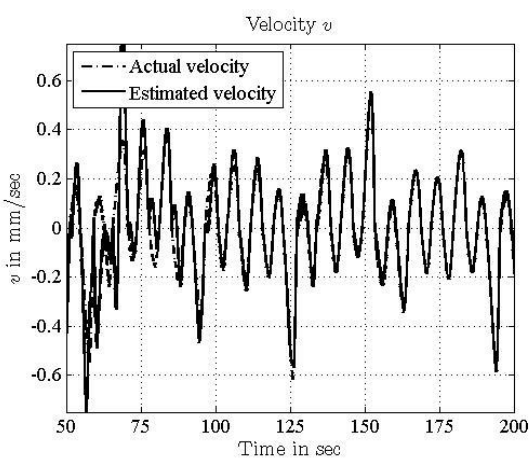

Fig. 2. Velocity of the low speed motion dynamics.

4.2. Example 2 – Nonlinear Mass-Spring-Damper (MSD) Model

We consider the MSD model introduced in (Stamnes *et al.*, 2009)

$$M\dot{v}+b|v|v+kp=u+w \tag{44}$$

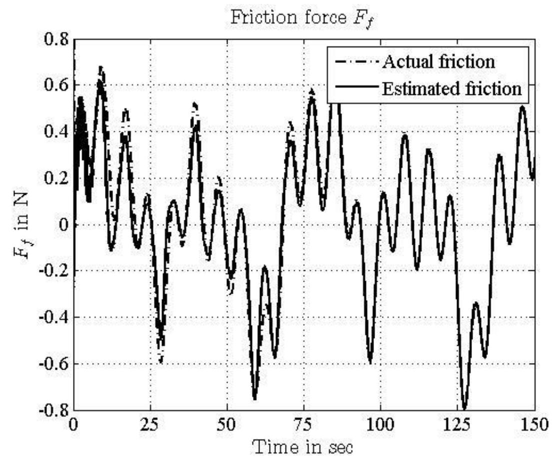

Fig. 3. Friction of the low speed motion dynamics.

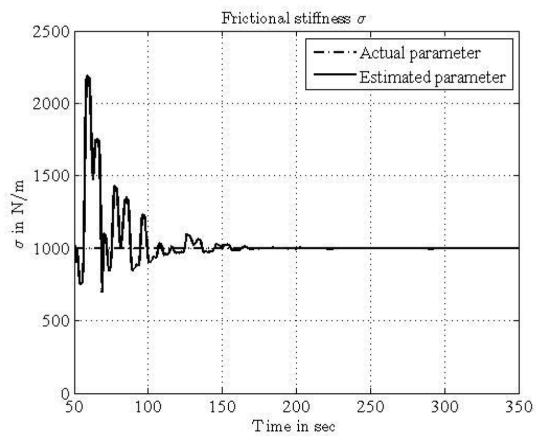

Fig. 4. Frictional stiffness of the low speed motion dynamics.

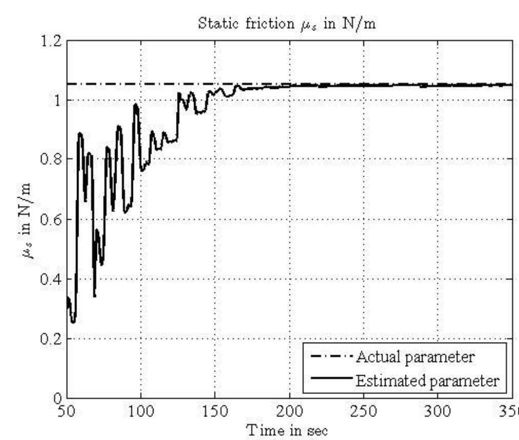

Fig. 5. Static friction of the low speed motion dynamics.

where u is the applied force, p the position and $v = \dot{p}$ the velocity, w a load disturbance. The positive constants k, M and b are unknown and denote the spring stiffness, the mass, and the nonlinear damping coefficient, respectively. We assume the position and the velocity measurements are both available but with some noise affecting the velocity signal. By assuming the fact that the load exhibit some noisy disturbance with similar frequency spectra, for simplicity we consider that both load and measurement noises have the same magnitude.

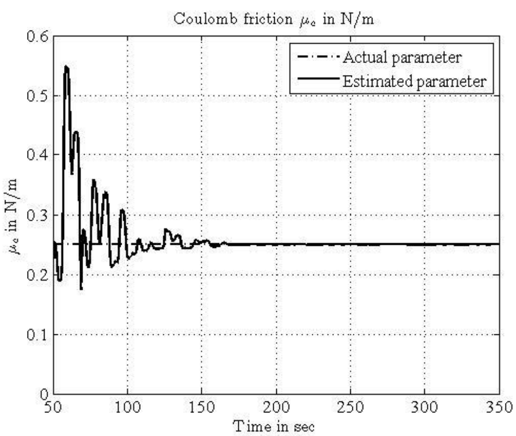

Fig. 6. Coulomb friction of the low speed motion dynamics.

Using the states $x_1 = p$ and $x_2 = v$ and the output vector components $y_1 = x_1$ and $y_2 = x_2 + w$, this dynamic model can be written in the state space representation (1) and (2) using the model matrices shown in Table 2. The components of the unknown vector θ are $\theta_1 = \frac{k}{M}$, $\theta_2 = \frac{b}{M}$ and $\theta_3 = \frac{1}{M}$, respectively (Stamnes *et al.*, 2009). The parameters characterizing the simulated MSD dynamics are chosen $M = 1\text{kg}$, $K = 0.5\text{Nm}^{-1}$ and $b = 0.5\text{Nsm}^{-1}$, respectively. To estimate the unknown states and parameters of the dynamics, we consider first the nominal case by assuming a disturbance free dynamics and we apply the NLAO scheme. Then, we consider the perturbed dynamics for which we apply both the NLAO and RNAO and compare their effectiveness and performances. The observer and adaptation law parameters of the NLAO (6)-(8) and the RNAO (24)-(27) are obtained in Table 2. Consider the input signal $u = 20\sin(2\pi t) + 10\sin\left(\pi t + \frac{5\pi}{2}\right)$ which results in sufficiently rich input signal that guarantees the fulfillment of the PE condition and that is necessary to ensure the convergence of the unknown parameters to their true values). The simulation results are shown in Figures 7-11. Both the state estimation error and the parameter error converge to zero quickly and accurately. Curves of the NLAO and RNAO designs overlap almost during all the time showing very similar results in terms of the position and velocity estimates (see Figures 7 and 8). In addition, curves of the NLAO, applied to the disturbance free system, and the RNAO, applied to the perturbed one, overlap almost during all the time in terms of parameter estimate dynamics (see Figures 9-11).

Table 2. Plant, NLAO and RNAO parameters in example 2.

Plant	$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = D = E = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C = I_2$
NALO	$L = \begin{pmatrix} 110.2 \\ 901.2 \end{pmatrix}$, $P = \begin{pmatrix} 191.0 & -1.0 \\ -1.0 & 0.1 \end{pmatrix}$, $\Phi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\Psi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\Gamma = \text{diag}(10^3, 10^2, 10)$
RNAO	$\bar{A} = A$, $\bar{C} = T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $\bar{D} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $G = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\Psi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 191.0 & -1.0 \\ -1.0 & 0.1 \end{pmatrix}$, $L = \begin{pmatrix} 55.1 & 55.1 \\ 448.1 & 453.1 \end{pmatrix}$, $\Gamma = \text{diag}(10^3, 10^2, 10)$

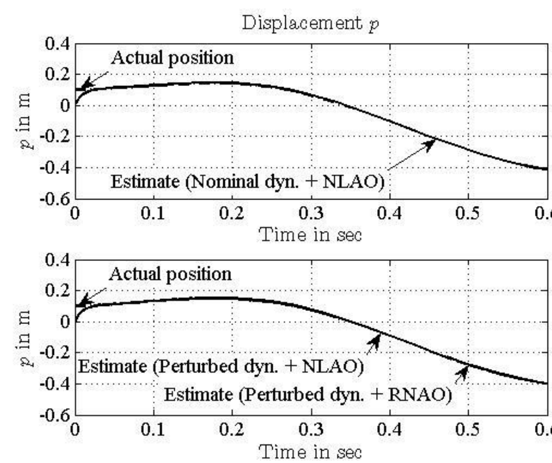

Fig. 7. Position of the MSD dynamics - Case of nominal system (top) and Case of perturbed system (bottom).

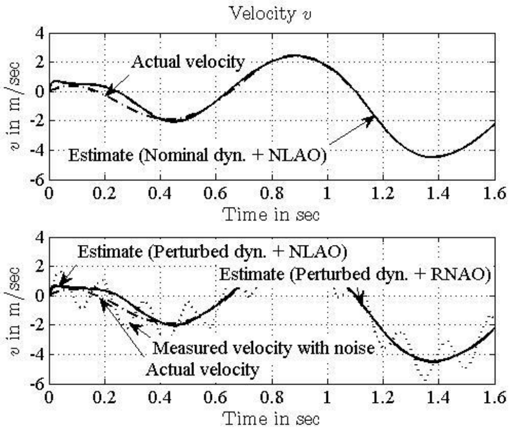

Fig. 8. Velocity of the MSD dynamics - Case of nominal system (top) and Case of perturbed system (bottom).

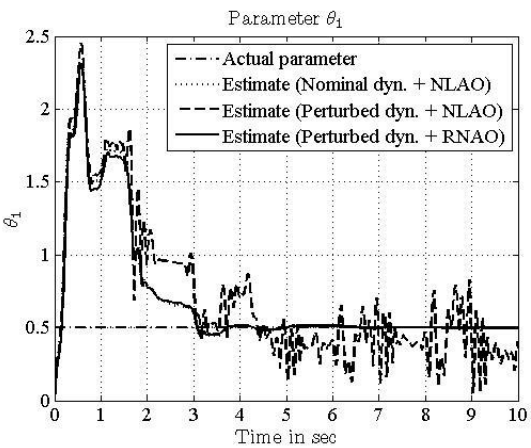

Fig. 9. Parameter θ_1 of the MSD dynamics.

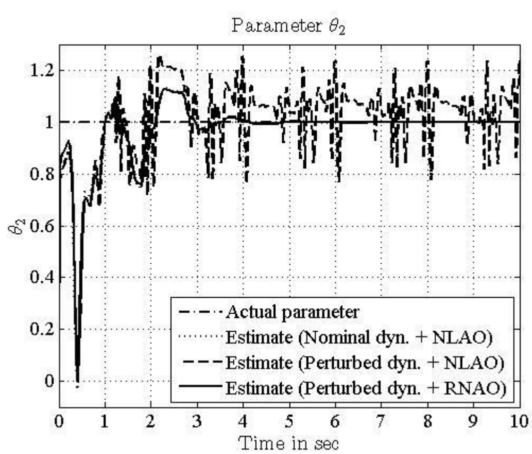

Fig. 10. Parameter θ_2 of the MSD dynamics.

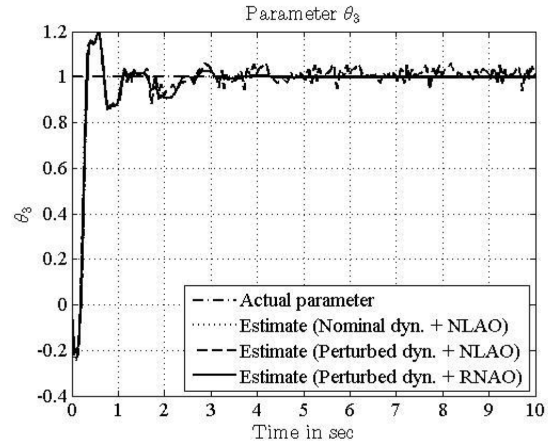

Fig. 11. Parameter θ_3 of the MSD dynamics.

4.3. Example 3 – Nonlinear Third Order Dynamics

Consider the dynamics below

$$\dot{x}_1 = x_2 + 10x_3 + x_1x_2 + bx_1x_3 \tag{45}$$

$$\dot{x}_2 = 2x_3 + bx_1x_3 \tag{46}$$

$$\dot{x}_3 = -4x_3 + \theta_1 u - \theta_2 x_2^{\,2} - \theta_3 x_3 \sqrt{1 + x_2^{\,2}} \tag{47}$$

$$y_1 = x_1 \quad \text{and} \quad y_2 = x_2 \tag{48}$$

y_1 and y_2 are the measurable states. The vector of the unknown parameters is $\theta = (\theta_1 \quad \theta_2 \quad \theta_3)^T$. For simulation, we choose $\theta_1 = 1$, $\theta_2 = 2$ and $\theta_3 = 3$. First, considering the nominal dynamics, the parameter b is assumed to be a well-posed constant $b = 1$. Thereafter, we simulate with some dynamic perturbation by assuming b uncertain. For simulation, we consider $b = \sin(2.75\pi t + \pi/3)$. The observer and adaptation law matrices of the NLAO and RNAO are shown in Table 3.

Using a linear combination of sine waves as an input signal, all results of the estimates of the states and parameters obtained with the two proposed methods (NLAO and RNAO) are shown in Figures 12-17. Both the state estimation error and the parameter error converge to zero quickly and accurately. The NLAO design has a better performance with the nominal case than with the perturbed one, while the RNAO converge to zero without shattering despite the presence of uncertain dynamics.

Table 3. Plant, NLAO and RNAO parameters in example 3.

NALO	$\Phi=\begin{pmatrix}1&0\\0&1\\0&-1/8\end{pmatrix}$, $\Psi=\begin{pmatrix}0&0\\0&0\\1/8&-1/8\end{pmatrix}$, $\Gamma=600I_3$, $P=\begin{pmatrix}2.8&0.1&-0.7\\0.1&2.2&-0.1\\-0.7&-0.1&1.6\end{pmatrix}$, $L=\begin{pmatrix}18.5&-0.1\\2.9&14.9\\17.3&2.7\end{pmatrix}$
RNAO	$\bar{A}=\begin{pmatrix}0&1&10\\0&0&2\\0&0&-4\end{pmatrix}$, $\bar{C}=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$, $G_0=\begin{pmatrix}5/4\\1/4\\0\end{pmatrix}$, $G_1=0_{31}$, $G=\begin{pmatrix}0\\0\\1\end{pmatrix}$, $\bar{D}=\begin{pmatrix}-1/4&5/4\\-1/4&5/4\\0&0\end{pmatrix}$, $E=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $K=\begin{pmatrix}-1/4&3/2\\0&1/4\\0&0\end{pmatrix}$, $T=I_2$, $\Phi=\begin{pmatrix}1&-1/20\\0&4/5\\0&-1/10\end{pmatrix}$, $\Psi=\begin{pmatrix}1/20&-1/4\\1/5&-1\\1/10&0\end{pmatrix}$, $P=\begin{pmatrix}2.8&0.1&-0.7\\0.1&2.2&-0.1\\-0.7&-0.1&1.6\end{pmatrix}$, $L=\begin{pmatrix}18.5&-0.1\\2.9&14.9\\17.3&2.7\end{pmatrix}$, $\Gamma=600I_3$

5. Conclusion

A new nonlinear adaptive observer and a corresponding robust scheme are derived for a wide class of nonlinear dynamic systems with unknown parameters, uncertain dynamics and disturbances. The asymptotic stability is developed using LMI frameworks.

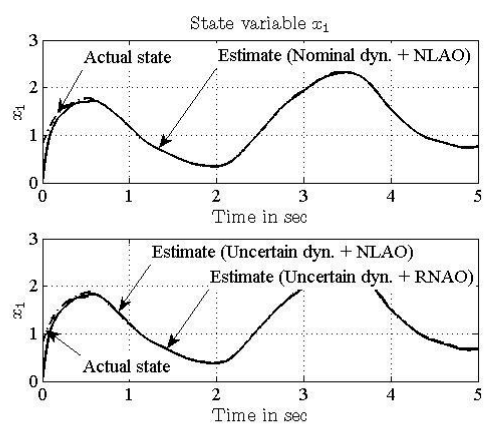

Fig. 12. State x_1 of the 3rd order dynamics (example 3) - Case of nominal system (top) and Case of uncertain system (bottom).

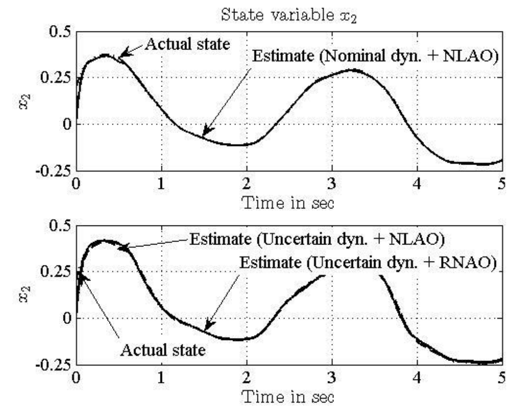

Fig. 13. State x_2 of the 3rd order dynamics (example 3) - Case of nominal system (top) and Case of uncertain system (bottom).

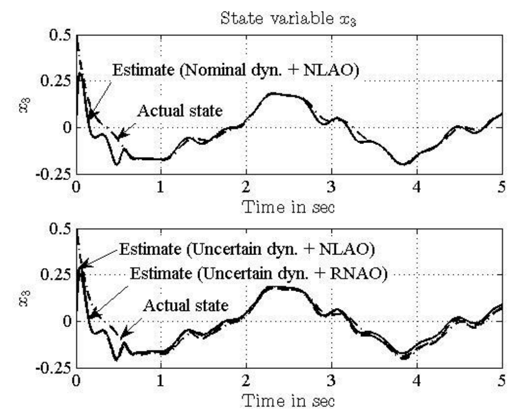

Fig. 14. State x_3 of the 3rd order dynamics (example 3) - Case of nominal system (top) and Case of uncertain system (bottom).

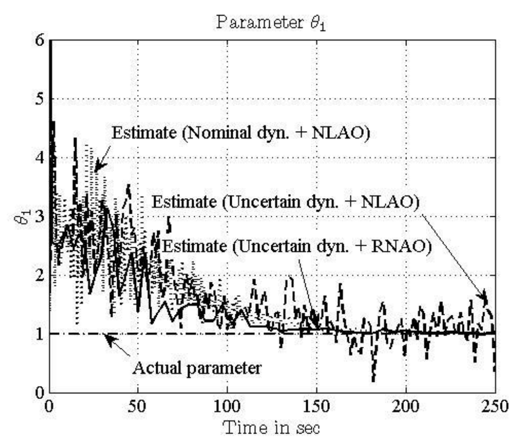

Fig. 15. Parameter θ_1 of the 3rd order dynamics (example 3).

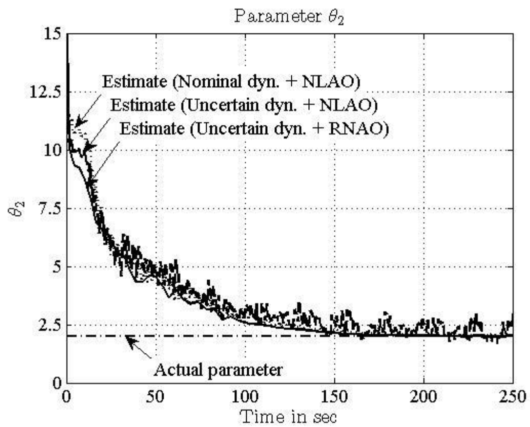

Fig. 16. Parameter θ_2 of the 3rd order dynamics (example 3).

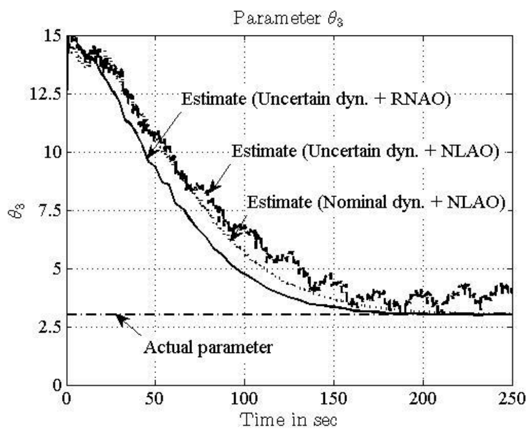

Fig. 17. Parameter θ_3 of the 3rd order dynamics (example 3).

The proposed LMI-based adaptive observers are designed for general nonlinear systems with unmeasured regressor matrices where the matching condition in terms of equality constraints on the Lyapunov matrix required in several works (*e.g.* (Liu, 2009; Dong & Mei, 2007; Marino, et al., 2001)) is not required. So, this method reduces the conservatism of this condition. The proposed estimation design exhibits a satisfactory convergence of both the states and the parameters to the actual values. Examples with simulation results successfully demonstrate the effectiveness of the proposed schemes. It is shown that both RNAO and NLAO track the trajectory but with a difference. In fact, the NLAO design has better performance with the nominal case than the perturbed one. We depict the difference that the state estimations under the RNAO approach the actual state coincidently while the NLAO does not. Moreover, the parameter estimation errors under the RNAO converge to zero despite the presence of uncertain dynamics, but the errors under the NLAO converge to zero only when the system is nominal. A recent work investigating the potential of the proposed method with an exponential stability of both the state and parameter estimates has been extended. Furthermore, experiments will be implemented to validate all those techniques within real-world scenarios.

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